SOLUTION OF AN ELASTIC STATIC PLANE PROBLEM FOR NONHOMOGENEOUS ISOTROPIC BODIES BY MEANS OF THE THEORY OF COMPLEX VARIABLES

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Real bodies can possess an initial nonhomogeneity due to an inclusion of a foreign material or imperfections, or as a result of being a composite material. The nonhomogeneity can be also generated by certain external fields and above all by a thermal field. It is known that operators in the constitutive equations describing viscoelastic materials contain parameters extremely sensitive to the change in temperature. In the case of the nonhomogeneous thermal field these parameters depend upon the space coordinate. The effect of induced nonhomogeneity on the stress distribution, caused by external forces, is much more pronounced and of longer duration than the effect of thermal stresses themselves [1]. Thus, the neglect of the former effect leads, in even simple situations, to physically inadmissible solutions.

Several papers have been devoted to the investigation of nonhomogeneous elastic bodies. For example in [2 to 4] an approximate hypothesis is assumed that one elastic modulus is varying while the Poisson ratio is kept constant. In other papers [5] the bodies are considered as being composed of layers of homogeneous elastic regions.

Misicu [6] and Misicu and Teodosiu [7] derived formulas for the complex mapping of stresses and displacement, valid for elastic and viscoelastic solids, with continuous nonhomogeneity of a general type in the case of plane and axisymmetric problems. In the present paper the method of solution of the elastic static plane problem for nonhomogeneous bodies is presented. This method is based on the mapping of the Kolosov [8 and 9] and Muskhelishvili [10] type; and conformal mapping (*). It is shown that to get a solution for the region with nonhomogeneity of a general type it is necessary to know the solution of the same problem for homogeneous medium.

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*) It will be shown in the following (Section 1) that a quasi-static problem for viscoelastic bodies in the presence of a stationary thermal field can be formally reduced by means of the Laplace transformation to the static elastic problem for the nonhomogeneous body, and the latter problem may be treated using the method developed in the present paper.

1. Basic equations and formulations of boundary value problems. Let us consider equations of quasi-static equilibrium, geometrical relations and constitutive equations for nonhomogeneous viscoelastic medium extending over the domain R

$$\sigma_{ij,j} + X_i = 0, \quad \varepsilon_{ij} = {}^1/_2(u_{i,j} + u_{j,i}), \quad s_{ij} = e_{ij} * dG_1, \quad \sigma_{kk} = (\varepsilon_{kk} - 3\alpha T) * dG_2 \tag{1.1}$$

$$s_{ij} = \sigma_{ij} - {}^1/_3\sigma_{kk}\delta_{ij}, \quad e_{ij} = \varepsilon_{ij} - {}^1/_3\varepsilon_{kk}\delta_{ij} \tag{1.2}$$

where σ_{ij} denotes the stress tensor, ε_{ij} is the strain tensor, u_i denotes components of elastic displacement, X_i denotes the body forces. The temperature $T = T(x_i)$ is stationary at the point $(x_i) = (x_1, x_2, x_3)$ and is measured relatively to the natural state of the body. The coefficient $\alpha = \alpha(x_i)$ is a material constant, $G_1 = G_1(x_i, t)$, and $G_2 = G_2(x_i, t)$ are functions describing the viscoelastic properties of the medium. By $*$ is denoted the convolution multiplication of the Stieltjes type of the corresponding functions (*).

Let us assume that $G_1, G_2, \sigma_{ij}, X_i, \varepsilon_{ij}$ (*f*) belong to the class H^1 and are of the order $O[\exp(p_0 t)]$ when $t \to \infty$ for $(x_i) \in R$, where p_0 is an arbitrary real constant. Equations (1.1) after the Laplace transformation take the form

$$\sigma_{ij,j}^{*} + X_i^* = 0, \quad \varepsilon_{ij}^* = {}^1/_2(u_{i,j}^{*} + u_{j,i}^{*}) \qquad \left(f^*(x_i, p) = \int_0^\infty e^{-pt} f(x_i, t)\, dt, \ \mathrm{Re}\, p > p_0\right) \tag{1.3}$$

$$s_{ij}^* = pG_1^* e_{ij}^{*}, \quad \sigma_{kk}^{*} = pG_2^*(\varepsilon_{kk}^{*} - 3\alpha T^*)$$

On introducing the notation $\sigma_{11} = \sigma_x$, $\sigma_{12} = \tau_{xy}, \ldots; \varepsilon_{11} = \varepsilon_x$, $\varepsilon_{12} = \varepsilon_{xy}, \ldots;$ $u_1 = u$, $u_2 = v$, Equations (1.2) and (1.3) in this case of a plane problem yield

$$\frac{\partial \sigma_x^*}{\partial x} + \frac{\partial \tau_{xy}^{*}}{\partial y} + X^* = 0, \quad \frac{\partial \tau_{xy}^{*}}{\partial x} + \frac{\partial \sigma_y^*}{\partial y} + Y^* = 0 \tag{1.4}$$

$$\varepsilon_x^* = \frac{\partial u^*}{\partial x}, \quad \varepsilon_y^* = \frac{\partial v^*}{\partial y}, \quad \varepsilon_{xy}^{*} = \frac{1}{2}\left(\frac{\partial u^*}{\partial y} + \frac{\partial v^*}{\partial x}\right) \tag{1.5}$$

$$\sigma_x^* = \lambda^*(\varepsilon_x^* + \varepsilon_y^*) + 2\mu^*\varepsilon_x^* - k^*T^*, \quad \sigma_y^* = \lambda^*(\varepsilon_x^* + \varepsilon_y^*) + 2\mu^*\varepsilon_y^* - k^*T^* \tag{1.6}$$

$$\tau_{xy}^{*} = 2\mu^*\varepsilon_{xy}^{*}, \quad \tau_{yz}^{*} = \tau_{zx}^{*} = 0 \tag{1.7}$$

where for the plane strain

$$\lambda^* = {}^1/_3 p\,(G_2^* - G_1^*), \qquad 2\mu^* = pG_1^*, \qquad k^* = p\alpha G_2^* \tag{1.8}$$

$$\sigma_z^{*} = \frac{G_2^* - G_1^*}{2G_2^* + G_1^*}(\sigma_x^* + \sigma_y^*) - \frac{3G_1^* G_2^* p\alpha T^*}{2G_2^* + G_1^*}, \qquad \varepsilon_z^* = 0 \tag{1.9}$$

and for the plane stress

$$\lambda^* = \frac{G_1^*(G_2^* - G_1^*)\,p}{2G_1^* + G_2^*}, \qquad 2\mu^* = pG_1^*, \qquad k^* = \frac{3G_1^* G_2^* p\alpha}{2G_1^* + G_2^*} \tag{1.10}$$

$$\varepsilon_z^* = \frac{G_1^* - G_2^*}{2G_1^* + G_2^*}(\varepsilon_x^* + \varepsilon_y^*) + \frac{3G_2^*}{2G_1^* + G_2^*}\alpha T^*, \qquad \sigma_z^* = 0 \tag{1.11}$$

Equations (1.4) to (1.7) are the same as those describing the plane problem for nonhomogeneous elastic solids [7]. In the following for the sake of simplicity, the asterisks are dropped.

We assume further that ε_x, ε_y, ε_{xy} and consequently σ_x, σ_y and τ_{xy} are uniform and continuous functions together with their first and second derivatives in the domain D occupied by the elastic body. Similarly $X = X(x, y)$

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*) The constitutive equations considered in this paper are of the relaxation type. Integral constitutive equations of the creeping type or differential relations, can be treated in a similar way. The notation used throughout this Section can be found in [11].

and $Y = Y(x,y)$ are analytical functions of x and y in the simply connected domain D_+ which fully contains the domain D. Equations (1.4) can be also written in the form

$$\frac{\partial}{\partial \bar{z}}(\sigma_y - \sigma_x + 2i\tau_{xy}) - \frac{\partial}{\partial z}(\sigma_x + \sigma_y) = X - iY \tag{1.12}$$

$$\left(\begin{matrix} z = x + iy, \\ \bar{z} = x - iy, \end{matrix} \quad \frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\right)$$

Equation (1.12) is satisfied identically if we assume

$$\sigma_x + \sigma_y = 4\frac{\partial^2 F}{\partial z\, \partial \bar{z}}, \qquad \sigma_y - \sigma_x + 2i\tau_{xy} = 4\frac{\partial^2 F}{\partial z^2} - M(z, \bar{z}) \tag{1.13}$$

where $F(z,\bar{z})$ is an analytic real-valued function of z and $\bar{z}$ in the domain $(D, \bar{D})$ such that its first four partial derivatives are continuous (*), and the function

$$-M(z, \bar{z}) = \int_0^{\bar{z}} \left[X\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right) - iY\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right)\right] d\bar{z} \tag{1.14}$$

is analytical in the domain $(D_+, \bar{D}_+)$.

Relations between the components of stress tensor σ_x, σ_y, τ_{xy} and displacements u and v, derived form (1.5) and (1.6) have the form

$$\frac{\partial \bar{U}}{\partial z} = -\frac{1}{4\mu}(\sigma_y - \sigma_x + 2i\tau_{xy}) = -\frac{1}{\mu}\frac{\partial^2 F}{\partial z^2} + \frac{M}{4\mu} \qquad \left(\begin{matrix} U = u + iv \\ \bar{U} = u - iv \end{matrix}\right) \tag{1.15}$$

$$\frac{\partial U}{\partial z} + \frac{\partial \bar{U}}{\partial \bar{z}} = \frac{\varkappa - 1}{4\mu}(\sigma_x + \sigma_y + 2kT) = \frac{\varkappa - 1}{\mu}\frac{\partial^2 F}{\partial z \partial \bar{z}} + \frac{k(\varkappa - 1)}{2\mu}T \qquad \left(\varkappa = \frac{\lambda + 3\mu}{\lambda + \mu}\right) \tag{1.16}$$

By elimination of U from (1.15) we can obtain a compatibility equation. We obtain the condition (1.17)

$$\frac{\partial^2}{\partial \bar{z}^2}\left(\frac{1}{\mu}\frac{\partial^2 F}{\partial z^2} - \frac{M}{4\mu}\right) + \frac{\partial^2}{\partial z^2}\left(\frac{1}{\mu}\frac{\partial^2 F}{\partial \bar{z}^2} - \frac{\bar{M}}{4\mu}\right) + \frac{\partial^2}{\partial z \partial \bar{z}}\left[\frac{\varkappa - 1}{\mu}\frac{\partial^2 F}{\partial z \partial \bar{z}} + \frac{k(\varkappa - 1)}{2\mu}T\right] = 0$$

which can be also written in the form

$$\frac{\partial^4 F}{\partial z^2 \partial \bar{z}^2} + A_1\frac{\partial^3 F}{\partial z \partial \bar{z}^2} + \bar{A}_1\frac{\partial^3 F}{\partial z^2 \partial \bar{z}} + A_2\frac{\partial^2 F}{\partial \bar{z}^2} + \bar{A}_2\frac{\partial^2 F}{\partial z^2} + A_3\frac{\partial^2 F}{\partial z\, \partial \bar{z}} = f(z, \bar{z}) \tag{1.18}$$

where

$$f(z, \bar{z}) = \frac{\mu}{\varkappa + 1}\left\{\frac{\partial^2}{\partial \bar{z}^2}\frac{M}{4\mu} + \frac{\partial^2}{\partial z^2}\frac{\bar{M}}{4\mu} - \frac{\partial^2}{\partial z\, \partial \bar{z}}\left(\frac{k(\varkappa - 1)}{2\mu}T\right)\right\}$$

$$A_1 = \frac{\partial}{\partial z}\ln\frac{\varkappa + 1}{\mu}, \qquad A_2 = \frac{\mu}{\varkappa + 1}\frac{\partial^2}{\partial z^2}\frac{1}{\mu}, \qquad A_3 = \frac{\mu}{\varkappa + 1}\frac{\partial^2}{\partial z\, \partial \bar{z}} \tag{1.19}$$

We assume that $A_i(z, \bar{z})$ and $f(z, \bar{z})$ are analytical functions of z and $\bar{z}$ in the domain $(D, \bar{D})$. It can be proved that an arbitrary solution of Equation (1.18) which has continuous partial derivatives up to the fourth order, should be an analytic function of z and $\bar{z}$ in this domain. Hence, the hypothesis concerning the continuity of stress components and their first and second derivatives in D includes in fact their analyticity in D. This statement generalizes the result of Muskhelishvili, ([10], Section 32), concerning homogeneous bodies.

It follows from (1.13) that the state of stress depends not directly upon F but through its second partial derivative. Denoting for example

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*) By $\bar{D}$ and $\bar{D}_+$ are denoted domains symmetric, respectively, to the domains D and D_+ relatively to the real axis. It is assumed that the origin belongs to the domain D.

$$2\frac{\partial F(z,\bar z)}{\partial \bar z} \equiv G(z,\bar z) \tag{1.20}$$

Equation (1.18) can be expressed as

$$\frac{\partial^3 G}{\partial z^2\,\partial \bar z} + \text{Re}\left(2A_1\,\frac{\partial^2 G}{\partial z\partial \bar z} + 2A_2\frac{\partial G}{\partial \bar z} + A_3\,\frac{\partial G}{\partial z}\right) = 2f(z,\bar z) \tag{1.21}$$

or in an equivalent form

$$\frac{\partial^3 G}{\partial z^2\,\partial \bar z} + \text{Re}\left[\frac{\partial^2 (B_1 G)}{\partial z\partial \bar z} + \frac{\partial (B_2 G)}{\partial z} + \frac{\partial (B_3 G)}{\partial \bar z} + B_4 G\right] = 2f(z,\bar z) \tag{1.22}$$

where (1.23)

$$B_1 = 2A_1, \qquad B_2 = A_3 - \frac{\partial A_1}{\partial \bar z}, \qquad B_3 = 2\left(A_2 - \frac{\partial A_1}{\partial z}\right), \qquad B_4 = 2\,\frac{\partial^2 A_1}{\partial z\partial \bar z} - 2\frac{\partial A_2}{\partial \bar z} - \frac{\partial A_3}{\partial z}$$

In the notation of Equation (1.20) the relations (1.13) take the form

$$\sigma_x + \sigma_y = 2\frac{\partial G}{\partial z}, \qquad \sigma_y - \sigma_x + 2i\tau_{xy} = 2\,\frac{\partial \bar G}{\partial z} - M(z,\bar z) \tag{1.24}$$

Equation (1.22) can be rewritten thus

$$\frac{\partial^3}{\partial z^2\,\partial \bar z}\left[G(z,\bar z) + IG(z,\bar z) - F_0(z,\bar z)\right] = 0, \qquad F_0(z,\bar z) \equiv 2\int_0^z dz\int_0^z dz\int_0^{\bar z} f(z,\bar z)\,d\bar z$$

$$IG(z,\bar z) \equiv \int_0^z \text{Re}\left[B_1 G + \int_0^{\bar z} B_2 G\,d\bar z + \int_0^z B_3 G\,dz + \int_0^z dz\int_0^{\bar z} B_4 G\,d\bar z\right]dz \tag{1.25}$$

From (1.25) we obtain

$$G(z,\bar z) + IG(z,\bar z) - F_0(z,\bar z) = \varphi(z) + z\overline{\varphi_1(z)} + \overline{\psi(z)} \tag{1.26}$$

where $\varphi(z)$, $\varphi_1(z)$ and $\psi(z)$ are arbitrary functions, holomorphic in D.

Equations (1.19), (1.20) and (1.26) imply that the derivative of function $G(z,\bar z) + IG(z,\bar z) - F_0(z,\bar z)$ with respect to z should be a real-valued function. By imposing a condition that also z-derivative of the function $\varphi(z) + z\overline{\varphi_1(z)} + \overline{\psi(z)}$ be real function, we obtain $\varphi_1(z) \equiv \varphi'(z)$ and consequently (1.26) becomes

$$G(z,\bar z) + IG(z,\bar z) = \varphi(z) + z\overline{\varphi'(z)} + \overline{\psi(z)} + F_0(z,\bar z) \tag{1.27}$$

Let C be the boundary of the domain D described by Equation $t = t(s)$ where $t(s)$ is an affix of the point C corresponding to the curvilinear abscissa s measured from an arbitrary chosen point on C. Evidently, it is assumed that $t(s+l) = t(s)$ and $t(s_1) \neq t(s_2)$, if $0 < s_1 < s_2 < l$, where l is the length of C-curve.

In the case of the first fundamental boundary value problem the prescribed components of external stress $\sigma_{nx} = \sigma_{nx}(s)$ and $\sigma_{ny} = \sigma_{ny}(s)$, applied on the contour C are related to the values on the boundary by well known Formulas

$$\sigma_{nx} = \sigma_x\cos(n,x) + \tau_{xy}\cos(n,y), \qquad \sigma_{ny} = \tau_{xy}\cos(n,x) + \sigma_y\cos(n,y) \tag{1.28}$$

where n denotes the outward normal to the contour C.

Relation (1.28) can be rewritten in the form (*)

$$\sigma_{nx} + i\sigma_{ny} = (\sigma_x + i\tau_{xy})\,y'(s) - (\tau_{xy} + i\sigma_y)\,x'(s), \; t(s) = x(s) + iy(s) \tag{1.29}$$

From (1.24) we obtain

$$\tau_{xy} + i\sigma_y = i\left(\frac{\partial G}{\partial z} + \frac{\partial G}{\partial \bar z}\right) - i\overline{M}(\bar z, z), \qquad \sigma_x + i\tau_{xy} = \frac{\partial G}{\partial z} - \frac{\partial G}{\partial \bar z} + \overline{M}(\bar z, z)$$

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*) Denote by $f(s)$ or $f(t,\bar t)$ the limiting values of certain function $f(z,\bar z)$ continuous in $(D,\bar D)$ for $z\in D$, $z\to C$, $\bar z\in \bar D$, $\bar z\to \bar C$, and by $\dot f(s)$ the derivative of $f(s)$ with respect to s.

By substitution $\tau_{xy} + i\sigma_y$ in (1.29) and noting that

$$\frac{\partial G}{\partial z} t'(s) + \frac{\partial G}{\partial \bar{z}} \overline{t'(s)} = \frac{dG}{ds} \qquad (t(s) = x(s) + iy(s))$$

we get

$$\sigma_{nx} + i\sigma_{ny} = -i\frac{dG}{ds} + i\overline{M(s)}\,\overline{t'(s)}$$

After integration with respect to s , the latter equation becomes

$$G(s) = i\int_0^s (\sigma_{nx} + i\sigma_{ny})\,ds + \int_0^s \overline{M(s)}\,\overline{t'(s)}\,ds + c \equiv H(s) \quad (c = \text{const}) \qquad (1.30)$$

It follows from the relation (1.13) that if the components of stresses are prescribed, then the function $G(z,\bar{z})$ is determined to within a constant. Consequently, by choosing $c = 0$ in Equation (1.30) the function $G(z,\bar{z})$ becomes fully determined by the state of stress.

Thus, the solution of the first boundary value problem is reduced to the determination of the solution $G(z,\bar{z})$ satisfying Equations (1.21) or (1.27) together with boundary condition (1.30). After having solved this problem, the stress components can be found using (1.24).

To solve the second boundary value problem we use the formulation in displacements. On account of Formulas (1.15) equation of equilibrium (1.12) can be written in terms of displacement in the form

$$\frac{\partial}{\partial \bar{z}}\left[\frac{\mu}{\varkappa - 1}\left(\frac{\partial U}{\partial z} + \frac{\partial \bar{U}}{\partial \bar{z}}\right)\right] + \frac{\partial}{\partial z}\left(\mu\frac{\partial U}{\partial \bar{z}}\right) = P(z,\bar{z}), \qquad P(z,\bar{z}) \equiv \frac{1}{2}\frac{\partial}{\partial \bar{z}}(kT) - \frac{1}{4}(X + iY) \qquad (1.31)$$

This equation can be written in the form

$$\frac{\partial}{\partial \bar{z}}\left\{\frac{\varkappa}{\varkappa - 1}\frac{\partial(\mu U)}{\partial z} + \frac{1}{\varkappa - 1}\frac{\partial(\mu\bar{U})}{\partial \bar{z}} - \frac{1}{\varkappa - 1}\frac{\partial\mu}{\partial z}U - \frac{1}{\varkappa - 1}\frac{\partial\mu}{\partial \bar{z}}\bar{U} - \right.$$

$$\left. - \int_0^{\bar{z}}\left[\frac{\partial}{\partial z}\left(\frac{\partial\mu}{\partial \bar{z}}U\right) + P(z,\bar{z})\right]d\bar{z}\right\} = 0$$

it follows from the above

$$\frac{\varkappa}{\varkappa - 1}\frac{\partial(\mu U)}{\partial z} + \frac{1}{\varkappa - 1}\frac{\partial(\mu\bar{U})}{\partial \bar{z}} - \frac{1}{\varkappa - 1}\frac{\partial\mu}{\partial z}U - \frac{1}{\varkappa - 1}\frac{\partial\mu}{\partial \bar{z}}\bar{U} -$$

$$- \int_0^{\bar{z}}\left[\frac{\partial}{\partial z}\left(\frac{\partial\mu}{\partial \bar{z}}U\right) + P(z,\bar{z})\right]d\bar{z} = \varphi'(z) \qquad (1.32)$$

where $\varphi(z)$ is an arbitrary function holomorphic in D . By eliminating $\partial(\mu\bar{U})/\partial\bar{z}$ from Equation (1.32) and from its complex conjugate equation we obtain

$$(\varkappa + 1)\mu U(z,\bar{z}) + JU(z,\bar{z}) = \varkappa\varphi(z) - z\overline{\varphi'(z)} - \overline{\psi(z)} - \int_0^z \frac{\partial\varkappa}{\partial z}\varphi(z)\,\partial z + P_0(z,\bar{z})$$

$$P_0(z,\bar{z}) \equiv \int_0^z\left[\varkappa\int_0^{\bar{z}} P(z,\bar{z})\,d\bar{z} - \int_0^z \bar{P}(\bar{z},z)\,dz\right]dz \qquad (1.33)$$

$$JU(z,\bar{z}) = -\int_0^z\left[\left(\frac{\partial\mu}{\partial z} + \mu\frac{\partial\varkappa}{\partial z}\right)U + \frac{\partial\mu}{\partial \bar{z}}\bar{U} + \varkappa\int_0^{\bar{z}}\frac{\partial}{\partial z}\left(\frac{\partial\mu}{\partial \bar{z}}U\right)d\bar{z} - \right.$$

$$\left. - \int_0^z \frac{\partial}{\partial \bar{z}}\left(\frac{\partial\mu}{\partial z}\bar{U}\right)dz\right]dz \qquad (1.34)$$

where $\psi(z)$ is again an arbitrary function, holomorphic in D .

The solution of the second boundary value problem is finally reduced to the solution of Equation (1.31) or equivalent to it (1.33) satisfying condition $U(s) = u(s) + iv(s)$, where $u(s)$ and $v(s)$ are components of elastic displacement, given on C.

2. Application of conformal mapping. Let us assume that the simply connected domain D. with boundary C in the plane $z = x + iy$ is transformed by means of conformal mapping $z = \omega(\zeta)$ into the circle Δ with boundary Γ, described by an Equation $|\zeta| = 1$, in the plane $\zeta = \xi + i\eta$ where $\omega(0) = 0$.

Since $\omega(\zeta)$ is holomorphic in Δ, then

$$\frac{\partial}{\partial z} = \frac{1}{\omega'(\zeta)} \frac{\partial}{\partial \zeta}, \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{\overline{\omega'(\zeta)}} \frac{\partial}{\partial \bar{\zeta}}, \quad dz = \omega'(\zeta)\, d\zeta, \quad d\bar{z} = \overline{\omega'(\zeta)}\, d\bar{\zeta} \tag{2.1}$$

Consequently (1.27) and (1.25) become

$$G^\circ(\zeta, \bar{\zeta}) + J^\circ G^\circ(\zeta, \bar{\zeta}) = \varphi(\zeta) + \frac{\omega(\zeta)}{\overline{\omega'(\zeta)}} \overline{\varphi'(\zeta)} + \overline{\psi(\zeta)} + F^\circ_0(\zeta, \bar{\zeta}) \tag{2.2}$$

$$J^\circ G^\circ(\zeta, \bar{\zeta}) = \int_0^\zeta \omega'(\zeta) \operatorname{Re} \Big[B_1^\circ G^\circ + \int_0^{\bar{\zeta}} \overline{\omega'(\zeta)} B_2^\circ G^\circ d\bar{\zeta} + \int_0^\zeta \omega(\zeta) B_3^\circ G^\circ d\zeta + $$
$$+ \int_0^\zeta \omega'(\zeta)\, d\zeta \int_0^{\bar{\zeta}} \overline{\omega'(\zeta)} B_4^\circ G^\circ d\bar{\zeta} \Big] d\zeta \tag{2.3}$$

$$F_0^\circ(\zeta, \bar{\zeta}) = 2 \int_0^\zeta \omega'(\zeta)\, d\zeta \int_0^\zeta \omega'(\zeta)\, d\zeta \int_0^{\bar{\zeta}} \overline{\omega'(\zeta)}\, f[\omega(\zeta), \overline{\omega(\zeta)}]\, d\bar{\zeta} \tag{2.4}$$

where $\varphi(\zeta)$ and $\psi(\zeta)$ are arbitrary functions holomorphic in Δ and

$$G^\circ(\bar{\zeta}, \zeta) \equiv G[\omega(\zeta), \overline{\omega(\zeta)}], \quad B_i^\circ(\zeta, \bar{\zeta}) \equiv B_i[\omega(\zeta), \overline{\omega(\zeta)}] \quad (i = 1, 2, 3, 4) \tag{2.5}$$

The boundary condition (1.30) after mapping is

$$G^\circ(\sigma) = H^\circ(\sigma) \tag{2.6}$$

where $\sigma = e^{i\theta}$ denote the curvilinear abscissa on the circle Γ and the function $H^\circ(\sigma)$ is uniquely defined in $H(s)$ since there is one to one correspondence $t = \omega(\tau)$ between affices t on the contour C and τ on the contour Γ.

Substitution of (2.1) into (1.33) and (1.34) results in

$$(\varkappa + 1)\mu U^\circ(\zeta, \bar{\zeta}) + J^\circ U^\circ(\zeta, \bar{\zeta}) = \varkappa\varphi(\zeta) - \frac{\omega(\zeta)}{\overline{\omega'(\zeta)}} \overline{\varphi'(\zeta)} - \overline{\psi(\zeta)} -$$
$$- \int_0^\zeta \omega'(\zeta) \frac{\partial \varkappa}{\partial \zeta} \varphi(\zeta)\, d\zeta + P_0^\circ(\zeta, \bar{\zeta}) \tag{2.7}$$

$$J^\circ U^\circ(\zeta, \bar{\zeta}) = - \int_0^\zeta \Big[\Big(\frac{\partial \mu}{\partial \zeta} + \mu \frac{\partial \varkappa}{\partial \zeta} \Big) U^\circ + \frac{\omega'(\zeta)}{\overline{\omega'(\zeta)}} \frac{\partial \mu}{\partial \zeta} \bar{U}^\circ + \varkappa \int_0^{\bar{\zeta}} \frac{\partial}{\partial \zeta} \Big(\frac{\partial \mu}{\partial \bar{\zeta}} U^\circ \Big) d\bar{\zeta} -$$
$$- \frac{\omega'(\zeta)}{\overline{\omega'(\zeta)}} \int_0^\zeta \frac{\partial}{\partial \bar{\zeta}} \Big(\frac{\partial \mu}{\partial \zeta} \bar{U}^\circ \Big) d\zeta \Big] d\zeta, \qquad \begin{array}{l} U^\circ(\zeta, \bar{\zeta}) \equiv U[\omega(\zeta), \overline{\omega(\zeta)}] \\ P_0^\circ(\zeta, \bar{\zeta}) \equiv P_0[\omega(\zeta), \omega(\bar{\zeta})] \end{array} \tag{2.8}$$

The image of the boundary condition (1.35) is

$$U^\circ(\sigma) = u^\circ(\sigma) + iv^\circ(\sigma)$$

where $u^\circ(\sigma)$ and $v^\circ(\sigma)$ are uniquely determined by $u(s)$ and $v(s)$.

3. Method of successive approximations. The boundary value problems for nonhomogeneous bodies will be solved by means of the method of successive approximations. Assuming that the first approximation corresponds to the homogeneous body subjected to the same condition of loading, the subsequent iterations introduce corrections due to the nonhomogeneity. The solution of the first boundary value problem can be found from Equations (2.2) and (2.6) as follows:

$$G^\circ(\zeta, \bar{\zeta}) = \sum_{n=1}^{\infty} G_n^\circ(\zeta, \bar{\zeta}) \tag{3.1}$$

$$G_1^\circ(\zeta, \bar{\zeta}) = \varphi_1(\zeta) + \frac{\omega(\zeta)}{\overline{\omega'(\zeta)}} \overline{\varphi'(\zeta)} + \overline{\psi(\zeta)} + F_0^\circ(\zeta, \bar{\zeta}) \tag{3.2}$$

$$G_n^\circ(\zeta, \bar{\zeta}) = \varphi_n(\zeta) + \frac{\omega(\zeta)}{\overline{\omega'(\zeta)}} \overline{\varphi_n'(\zeta)} + \overline{\psi_n(\zeta)} - I^\circ G_{n-1}^\circ(\zeta, \bar{\zeta}) \quad (n \geqslant 2) \tag{3.3}$$

where functions $\varphi_n(\zeta)$ and $\psi_n(\zeta)$ $(n \geq 1)$ are holomorphic in Δ and should be determined from boundary conditions

$$\varphi_1(\tau) + \frac{\omega(\tau)}{\overline{\omega'(\tau)}} \overline{\varphi_1'(\tau)} + \overline{\psi_1(\tau)} = H^\circ(\tau) - F_0^\circ(\tau, \bar{\tau}) \tag{3.4}$$

$$\varphi_n(\tau) + \frac{\omega(\tau)}{\overline{\omega'(\tau)}} \overline{\varphi_n'(\tau)} + \overline{\psi_n(\tau)} = J^\circ G_{n-1}^\circ(\tau, \bar{\tau}) \quad (n \geqslant 2) \tag{3.5}$$

As is shown in Section 1, the elimination of an additive constant in the boundary conditions (1.30), in accordance with (2.6), permits to express $G^\circ(\zeta, \bar{\zeta})$ in terms of stress. However, since functions $\varphi(\zeta)$ and $\psi(\zeta)$ are not fully determined from (2.2) we can impose an additional condition ([10], Section 34)

$$\varphi(0) = 0, \qquad \operatorname{Im} \varphi'(0) = 0$$

This implies that in the considered scheme of solution we can assume

$$\varphi_n(0) = 0, \qquad \operatorname{Im} \varphi_n'(0) = 0 \qquad (n \geqslant 1) \tag{3.6}$$

A solution of the subsequent boundary value problems (3.4) and (3.5) can be achieved by means of methods known in the theory of elasticity for homogeneous bodies, i.e. method of expanding in power series, integral methods, etc.

An example concerning a solution of the first boundary value problem for a circle, based on the known Muskhelishvili's power series solution, will be given in Section 4.

The convergence of series (3.1) depends upon the conditions imposed on functions $H^\circ(\tau)$, $F_0^\circ(\xi, \bar{\xi})$, $\omega(\zeta)$, and also upon the type of functions describing the nonhomogeneous properties of the body.

A solution of the second boundary value problem can be found from (2.7) and (2.8), according to

$$U^\circ(\zeta, \bar{\zeta}) = \sum_{n=1}^{\infty} U_n^\circ(\zeta, \bar{\zeta}) \tag{3.7}$$

$$(\varkappa+1)\mu U_1^\circ(\zeta, \bar{\zeta}) = \varkappa\varphi_1(\zeta) - \frac{\omega(\zeta)}{\overline{\omega'(\zeta)}} \overline{\varphi_1'(\zeta)} - \overline{\psi_1(\zeta)} - \int_0^\zeta \omega'(\zeta) \frac{\partial \varkappa}{\partial \zeta} \varphi_1(\zeta)\, d\zeta + P_0^\circ(\zeta, \bar{\zeta})$$

$$(\varkappa+1)\mu U_n^\circ(\zeta, \bar{\zeta}) = \varkappa\varphi_n(\zeta) - \frac{\omega(\zeta)}{\overline{\omega'(\zeta)}} \overline{\varphi_n'(\zeta)} - \overline{\psi_n(\zeta)} - \tag{3.8}$$

$$-\int_0^\zeta \omega'(\zeta) \frac{\partial \varkappa}{\partial \zeta} \varphi_n(\zeta)\, d\zeta - J^\circ U_{n-1}^\circ(\zeta, \bar{\zeta}) \quad (n \geqslant 2) \tag{3.9}$$

where the functions $\varphi_n(\zeta)$ and $\psi_n(\zeta)$ $(n \geq 1)$, are holomorphic in Δ, and are determined from the boundary conditions

$$\varkappa\varphi_1(\tau) - \frac{\omega(\tau)}{\overline{\omega'(\tau)}} \overline{\varphi_1'(\tau)} - \overline{\psi_1(\tau)} - \int_0^\tau \omega'(\zeta) \frac{\partial \varkappa}{\partial \zeta} \varphi_1(\zeta)\, d\zeta = (\varkappa+1)\mu\,[u(\tau) + iv(\tau)] - P_0^\circ(\tau, \bar{\tau}) \tag{3.10}$$

$$\varkappa\varphi_n(\tau) - \frac{\omega(\tau)}{\overline{\omega'(\tau)}}\overline{\varphi_n'(\tau)} - \overline{\psi_n(\tau)} - \int_0^{\tau}\omega'(\zeta)\frac{\partial\varkappa}{\partial\zeta}\varphi_n(\zeta)\,d\zeta = J^\circ U^{\circ}_{n-1}(\tau,\bar\tau) \qquad (n \geqslant 2) \quad (3.11)$$

It is seen from (2.7) that the functions $\varphi(\zeta)$ and $\psi(\zeta)$ are not uniquely determined by $U^\circ(\tau, \bar\tau)$. However, in the considered scheme of solution we can add the following condition

$$\varphi_n(0) = 0 \qquad (n \geqslant 1) \qquad (3.12)$$

and then, both $\varphi_n(\zeta)$ and $\psi_n(\zeta)$ become fully determined by means of $U^\circ(\zeta,\bar\zeta)$.

The boundary value problem (3.10) and (3.11) differs from the common elastic problems for homogeneous bodies by the presence of an integral term. Assuming a frequently used hypothesis (*) that $\varkappa = \varkappa_0 =$ const , relations (3.8) to (3.11) become

$$(\varkappa_0 + 1)\mu U_1^\circ(\zeta,\bar\zeta) = \varkappa_0\varphi_1(\zeta) - \frac{\omega(\zeta)}{\overline{\omega'(\zeta)}}\overline{\varphi_1'(\zeta)} - \overline{\psi_1(\zeta)} + P_0^\circ(\zeta,\bar\zeta) \qquad (3.13)$$

$$(\varkappa_0 + 1)\mu U_n^\circ(\zeta,\bar\zeta) = \varkappa_0\varphi_n(\zeta) - \frac{\omega(\zeta)}{\overline{\omega'(\zeta)}}\overline{\varphi_n'(\zeta)} - \overline{\psi_n(\zeta)} - J^\circ U^{\bullet}_{n-1}(\zeta,\bar\zeta) \quad (n \geqslant 2) \qquad (3.14)$$

$$\varkappa_0\varphi_1(\tau) - \frac{\omega(\tau)}{\overline{\omega'(\tau)}}\overline{\varphi_1'(\tau)} - \overline{\psi_1(\tau)} = (\varkappa_0+1)\mu\,[u(\tau) + iv(\tau)] - P_0^\circ(\tau,\bar\tau) \qquad (3.15)$$

$$\varkappa_0\varphi_n(\tau) - \frac{\omega(\tau)}{\overline{\omega'(\tau)}}\overline{\varphi_n'(\tau)} - \overline{\psi_n(\tau)} = J^\circ U^{\bullet}_{n-1}(\tau,\bar\tau) \qquad (3.16)$$

and the solution of related boundary value problems (3.10) and (3.11) can be reached by means of methods used in the theory of elasticity for homogeneous bodies.

4. Numerical example. We shall solve the first boundary value problem for the domain D bounded by a circle $|z| = R$ and subjected on the circumference to a uniform radial tensile load of intensity p (Fig.1a). Applying a conformal mapping $z = R\zeta$, that domain D transforms into the domain Δ in the complex plane ζ representing a unit radius circle $|\zeta| = 1$, (Fig. 1b).

Let $z = re^{i\theta}$, $\zeta = \rho e^{i\theta}$ $(\rho = r/R)$. Introducing the polar components of stresses σ_r, σ_θ and $\tau_{r\theta}$, we obtain from (1.24) expressions determining these components corresponding to various stages of the iteration process

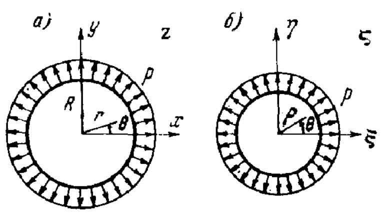

Fig. 1

$$\tau_{r\theta}^{(n)} = \frac{1}{R}\,\mathrm{Im}\left(\frac{\partial\overline{G}_n^\circ}{\partial\zeta}\,e^{2i\theta}\right)$$

$$\sigma_r^{(n)} = \frac{1}{R}\left[\frac{\partial\overline{G}_n^\circ}{\partial\zeta} - \mathrm{Re}\left(\frac{\partial\overline{G}_n^\circ}{\partial\zeta}\,e^{2i\theta}\right)\right] \qquad (4.1)$$

$$\sigma_\theta^{(n)} = \frac{1}{R}\left[\frac{\partial G_n^\circ}{\partial\zeta} + \mathrm{Re}\left(\frac{\partial\overline{G}_n^\circ}{\partial\zeta}\,e^{2i\theta}\right)\right]$$

If the components of stress applied to the contour are $\sigma_{n\xi} = p\cos\theta$, $\sigma_{n\eta} = p\sin\theta$, then taking into account that $d\sigma = Rd\theta$, we obtain from (1.30)

$$H^\circ(\theta) = iR\int_0^\theta(\sigma_{n\xi} + i\sigma_{n\eta})\,d\theta = ipR\int_0^\theta e^{i\theta}d\theta = pRe^{i\theta}$$

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*) This hypothesis can be assumed as a first approximation for an arbitrary nonhomogeneous body since $\varkappa$ depends solely upon the Poisson coefficient, which varies for all known materials within sufficiently narrow ranges.

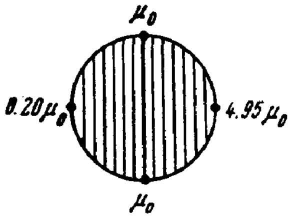

Fig. 2

Consider the nonhomogeneity of the type

$$\mu = \mu_0 \exp\left[\frac{\alpha}{R}(z + \bar{z})\right], \qquad \varkappa = \varkappa_0 .$$

where α and $\varkappa_0$ are dimensionless parameters, whereas μ_0 is of the same dimension as μ. The lines μ = const are then parallel to the η-axis, (Fig.2). From (1.19), (1.23) and (2.3) we get

$$J^\circ G^\circ(\zeta, \bar{\zeta}) = \int_0^\zeta \mathrm{Re}\left[-2\alpha G^\circ + \frac{(\varkappa_0 - 1)\alpha^2}{\varkappa_0 + 1}\int_0^{\bar{\zeta}} G^\circ d\bar{\zeta} + \frac{2\alpha^2}{\varkappa_0 + 1}\int_0^\zeta G^\circ d\zeta\right] d\zeta$$

Applying now the scheme of solution presented in Section 4 and also using the Muskhelishvili's solution of the first boundary value problem ([10], Section 54) we obtain expressions for the function $\Omega^*(\zeta,\bar{\zeta})$ and corresponding components of stress. The first three iterations of σ_r, σ_θ and $\tau_{r\theta}$, are

$$\sigma_r^{(1)} = \sigma_\theta^{(1)} = p, \qquad \tau_{r\theta}^{(1)} = 0$$

$$\sigma_r^{(2)} = \frac{p(\varkappa_0 - 1)\alpha^2}{2(\varkappa_0 + 1)}(1 - \rho^2), \quad \sigma_\theta^{(2)} = \frac{p(\varkappa_0 - 1)\alpha^2}{2(\varkappa_0 + 1)}(1 - 3\rho^2), \quad \tau_{r\theta}^{(2)} = 0$$

$$\sigma_r^{(3)} = \frac{p(\varkappa_0 - 1)\alpha^3}{24(\varkappa_0 + 1)^2}(1 - \rho^2)\left[2\alpha(\varkappa_0 - 1)(2 - \rho^2) + 8(\varkappa_0 + 1)\rho\cos\theta - \alpha(1 + \rho^2)\cos 2\theta\right]$$

$$\sigma_\theta^{(3)} = \frac{p(\varkappa_0 - 1)\alpha^3}{24(\varkappa_0 + 1)^2}\left[2\alpha(\varkappa_0 - 1)(2 - 9\rho^2 + 5\rho^4) + 8(\varkappa_0 + 1)(3 - 5\rho^2)\rho\cos\theta + \right.$$
$$\left. + \alpha(1 - 12\rho^2 + 15\rho^4)\cos 2\theta\right]$$

$$\tau_{r\theta}^{(3)} = \frac{p(\varkappa_0 - 1)\alpha^3}{24(\varkappa_0 + 1)^2}(1 - \rho^2)\left[8(\varkappa_0 + 1)\rho\sin\theta + \alpha(1 - 5\rho^2)\sin 2\theta\right]$$

Table 1 presents relative values of σ_r and σ_θ at $\theta = 0$ for $\varkappa_0 = 1.8$ and $\alpha = 0.8$, corresponding to the first three stages of iteration, so that in the Table 1

$$\sigma_{r1} = \frac{\sigma_r^{(1)}}{p}, \quad \sigma_{r2} = \frac{\sigma_r^{(1)} + \sigma_r^{(2)}}{p}, \quad \sigma_{r3} = \frac{\sigma_r^{(1)} + \sigma_r^{(2)} + \sigma_r^{(3)}}{p} \quad \left(\begin{array}{c}\sigma_{\theta1}, \sigma_{\theta2}, \sigma_{\theta3} \\ \text{analogically}\end{array}\right)$$

In this case μ changes from $0{,}20\mu_0$ to $4.95\,\mu_0$, (Fig.2). For $\theta = 0$ the shear stress $\tau_{r\theta}^{(n)} = 0$, $n \geqslant 1$, since the nonhomogeneity and load are symmetrical with respect to ξ-axis.

Table 1.

Stress	ρ					
	0.0	0.2	0.4	0.6	0.8	1.0
σ_{r1}	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
σ_{r2}	1.09143	1.08777	1.07680	1.05851	1.03291	1.00000
σ_{r3}	1.09526	1.10063	1.09578	1.07863	1.04728	1.00000
$\sigma_{\theta1}$	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
$\sigma_{\theta2}$	1.09143	1.08046	1.04754	0.99269	0.91589	0.81714
$\sigma_{\theta3}$	1.09874	1.11280	1.09068	1.02310	0.90253	0.72282

Results presented in Table 1 show that the presence of nonhomogeneity alters the state of stress both qualitatively and quantitatively. In the considered case the maximum value of radial and circumferential stresses σ_r and σ_θ changes respectively by + 10% and – 28%. Also, there appears the shear stress $\tau_{r\theta}$, although the loading is axisymmetric.

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